A note on commuting automorphisms of some finite p-groups

Sandeep Singh[*] **and Deepak Gumber**[†]
School of Mathematics and Computer Applications
Thapar University, Patiala - 147 004, India

An automorphism α of a group G is called a commuting automorphism if each element x in G commutes with its image $\alpha(x)$ under α. Let $A(G)$ denote the set of all commuting automorphisms of G. Rai [Proc. Japan Acad., Ser. A **91** (2015), no. 5, 57-60] has given some sufficient conditions on a finite p-group G such that $A(G)$ is a subgroup of $\mathrm{Aut}(G)$ and, as a consequence, has proved that in a finite p-group G of co-class 2, where p is an odd prime, $A(G)$ is a subgroup of $\mathrm{Aut}(G)$. We give here very elementary and short proofs of main results of Rai.

2010 Mathematics Subject Classification: 20F28, 20F18.

Keywords: Commuting automorphism, co-class 2 group.

Let G be a finite non-abelian p-group of order p^n, and let $\gamma_k(G)$ and $Z_k(G)$ respectively denote the kth terms of the lower and upper central series of G. For convenience, $\gamma_2(G)$ and $Z_1(G)$ are respectively denoted as G' and $Z(G)$. We call G an $A(G)$-group if $A(G)$ is a subgroup of $\mathrm{Aut}(G)$. Let $\alpha, \beta \in A(G)$. Then, for any $x \in G^2$, $x^{-1}\alpha(x), x^{-1}\beta(x) \in Z_2(G))$, by [1, Theorem 1.4]. Now if p is an odd prime, then $G^2 = G$ and hence $x^{-1}\alpha(x), x^{-1}\beta(x) \in Z_2(G)$. If $Z_2(G))$ is abelian, then $[\alpha(x), \beta(x)] = [x^{-1}\alpha(x), x^{-1}\beta(x)] = 1$ and hence $A(G)$ is a subgroup of $\mathrm{Aut}(G)$ by [1, Lemma 2.4(vi)]. Suppose that $|Z_2(G)/Z(G)| = p^2$ and $Z(G) = \gamma_k(G)$ for some $k \geq 2$. Then G is of nilpotence class k and hence $\gamma_{k-1}(G) \leq Z_2(G)$. If $k = 2$, then G is an $A(G)$-group by [3, Lemma 2.2]. Assume that $k \geq 3$. Since $\gamma_{k-1}(G)$ commutes with $Z_2(G)$, $\gamma_{k-1}(G)$ is a central subgroup of $Z_2(G)$. It follows that $Z_2(G)$ is abelian, because $Z_2(G)/\gamma_{k-1}(G)$ is cyclic. We have thus proved the following main theorem of Rai.

Theorem 1 ([2, Theorem 3.3]). *Let p be an odd prime and G be a finite p-group such that $|Z_2(G)/Z(G)| = p^2$ and $Z(G) = \gamma_k(G)$ for some $k \geq 2$. Then G is an $A(G)$-group.*

Observe that if G is of co-class 2, then $|Z_2(G)/Z(G)| = p$ or p^2. It follows that $Z_2(G)$ is abelian as explained above. We thus have the following theorem of Rai.

Theorem 2 ([2, Theorem A]). *Let G be a finite p-group of co-class 2 for an odd prime p. Then G is an $A(G)$-group.*

References

[1] M. Deaconescu, G. Silberberg and G. L. Walls, On commuting automorphisms of groups, Arch. Math. **79**(2002), no. 6, 479-484.

[2] P. K. Rai, On commuting automorphisms of finite p-groups, Proc. Japan Acad., Ser. A **91** (2015), no. 5, 57-60.

[3] F. Vosooghpour and M. Akhavan-Malayeri, On commuting automorphisms of p-groups, Comm. Algebra **41** (2013), no. 4, 1292-1299.

[*]Research supported by CSIR, New Delhi
[†]Research supported by NBHM, Department of Atomic Energy, Govt. of India